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On the non-steady motion of visco-plastic liquids in porous media

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The previous works on the motions of visco-plastic fluids in porous media are generalized in order to include the phenomenon of hysteresis. Only slowly varying motions are studied to any extent, when all inertial terms can be neglected without appreciable error. A general equation is deduced and some features of the motion are discussed. Simple particular cases without free or seepage surfaces are considered, when the fluid moves in the whole porous medium. A plane example points out the existence of different regions of motion during the variation of the heads in upper and lower water reservoirs. The necessity of further research on the flow of visco-plastic fluids through porous media is emphasized.

1. Introduction

The motion of Newtonian fluids in porous media is a problem that has been studied at length; no doubt it will continue to be so studied in the future. However, some researches have revealed that, at small and very small velocities, there are fluids which move, in microporous media, slower than is required by Darcy's law (e.g. Merkel 1956; Gheorghitza 1959). Thus arose the concept of 'initial gradient', which could describe satisfactorily the motion of these fluids (Polubarinova-Kotchina 1952, p. 28); therefore it was necessary to consider theoretically, as well as experimentally, the motion of these fluids. Five years ago a partial differential equation was given describing the motions with initial gradient in porous homogeneous media and it was emphasized that these motions are due to the rheological properties of fluids (Gheorghitza 1959). At the same time experimental investigations were made on the motion of fluids exhibiting the phenomenon of initial gradient, i.e. on visco-plastic fluids (e.g. Sultanov 1960).

It is well known that in capillary tubes the discharge of Newtonian liquids in steady flow is given by the Poiseuille formula, i.e. the discharge is proportional to the pressure gradient. This formula corresponds to Darcy's law according to which the velocity of the filtration liquid is a linear function of the pressure gradient and vanishes only when the pressure gradient is zero. Similarly, it was observed that the curve $\epsilon' = F(\sigma)$ of ideal Bingham bodies (Persoz 1960, p. 24) has the same shape as the curve $V = f(|\text{grad}\, h|)$ for some fluids moving in porous media, where V is the intensity of the filtration velocity,

$$h = p\gamma^{-1} + z, \qquad (1)$$

p being the pressure, γ the specific weight of the liquid and the Oz-axis chosen along the upward vertical; these fluids received the name of 'visco-plastic'

(Sultanov 1960). More properly, these fluids can be named 'ideal visco-plastic filtration fluids'. Later an attempt was made to describe the motion when the relation $V = f(|\text{grad}\, h|)$ is no longer linear; this case corresponds to real Bingham bodies (Gheorghitza 1961).

Here we consider slow motions without free or seepage surfaces of incompressible filtration fluids exhibiting hysteresis, as do some thixotropic or antithixotropic fluids.

2. Equations of motion

The equation of motion of ideal visco-plastic filtration fluids can be written in the form

$$\mathbf{V} + \frac{k}{mg}\frac{\partial \mathbf{V}}{\partial t} = -k\left(1 - \frac{K^*}{|\text{grad}\, h|}\right)\text{grad}\, h, \tag{2}$$

where k is the filtration coefficient, m the porosity, K^* the initial value of $|\text{grad}\, h|$, it being supposed that $|\text{grad}\, h| > K^*$ in the region of motion (Gheorghitza 1959); as is usual in underground hydrodynamics, it is assumed that from the total derivative $d\mathbf{V}/dt$ only the first term is of importance (Polubarinova-Kotchina 1952, pp. 36–7). Denoting $\text{grad}\, h$ by $\mathbf{J}$ and $\mathbf{J}_0$ the unit vector of $\mathbf{J}$ with $J = |\mathbf{J}|$, (2) becomes for steady motion

$$\mathbf{V} = F(J)\,\mathbf{J}_0, \tag{3}$$

where $F(J)$ is a certain function (Gheorghitza 1961). It is then assumed that $F(J)$ is the same function irrespective of the sign of dV/dt. Now these equations describing the motion of visco-plastic fluids will be extended by taking as the relation between V and J

$$\mathbf{V} + \frac{k(J)}{mg}\frac{\partial \mathbf{V}}{\partial t} = \begin{cases} 0 & \text{for } J \leqslant a_+, \\ -F_+(J)\,\mathbf{J}_0 & \text{for } J_{\max} > J \geqslant a_+, \end{cases} \tag{4}$$

when $dV/dt > 0$, and

$$\mathbf{V} + \frac{k(J)}{mg}\frac{\partial \mathbf{V}}{\partial t} = \begin{cases} -F_-(J)\,\mathbf{J}_0 & \text{for } J_{\max} > J \geqslant a_-, \\ 0 & \text{for } J \leqslant a_-, \end{cases} \tag{5}$$

when $dV/dt < 0$; F_+ and F_- are positive functions.

Here the inertial term is taken into account in the usual form (see, for example, Polubarinova-Kotchina 1952, pp. 36, 542). The positive function $k(J)$ is supposed to be known from experiment; nevertheless, the magnitude of $k(J)$ must be of the same order as that of the usual filtration coefficient. Then if we suppose that $\partial\mathbf{V}/\partial t$ is bounded by a sufficiently small constant and consider the values of the filtration coefficient divided by mg, we can disregard the inertial terms of (4) and (5); the proof for Newtonian filtration liquids is given, for instance, by Polubarinova-Kotchina (1952, pp. 36–7), and for visco-plastic filtration fluids the proof can be equally easily given, following the same lines. In this way we can take instead of (4) and (5) the relations

$$\mathbf{V} = \begin{cases} \left.\begin{array}{ll} 0 & \text{for } J \leqslant a_+, \\ -F_+(J)\,\mathbf{J}_0 & \text{for } J_{\max} > J \geqslant a_+, \end{array}\right\} \dfrac{dV}{dt} > 0, \\ \left.\begin{array}{ll} -F_-(J)\,\mathbf{J}_0 & \text{for } J_{\max} > J \geqslant a_-, \\ 0 & \text{for } J \leqslant a_-, \end{array}\right\} \dfrac{dV}{dt} < 0, \end{cases} \tag{6}$$

where F_+ and F_- have the meaning given earlier.

The relations (4), (5) and (6) express the fact that the connexion between V J is no longer reversible: if the motion starts after a critical initial point is surpassed and J *grows monotonically* till a certain value is reached and then immediately *decreases monotonically*, from 0 to $V_{\max}$ we have one relation between V and J, and from $V_{\max}$ back to 0 we have another relation between the same variables; i.e. $V = F_+(J)$ when $dV/dt > 0$ and $V = F_-(J)$ when $dV/dt < 0$. From the above formulation it follows that a_- is a certain function of $J_{\max}$ but no effort is made here to deduce theoretically this function.

Taking into account the continuity equation, $\operatorname{div} \mathbf{V} = 0$, we shall obtain from (6), denoting the Laplacian operator by Δ, the equation

$$2F(|\operatorname{grad} h|)\,|\operatorname{grad} h|\,\Delta h + \left[F'(|\operatorname{grad} h|) - \frac{F(|\operatorname{grad} h|)}{|\operatorname{grad} h|}\right] \operatorname{grad} h \,.\operatorname{grad}\,(|\operatorname{grad} h|^2) = 0, \tag{7}$$

in the region where the fluid is moving; F will have the index $+$ or $-$, but $F'(|\operatorname{grad} h|)$ means the derivative of F with respect to the variable $|\operatorname{grad} h|$.

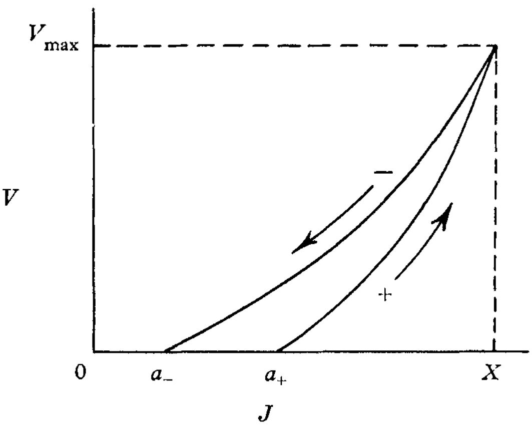

FIGURE 1. 'Loading' and 'unloading' loop for typical visco-plastic fluid in porous medium. J is the magnitude of the filtration pressure gradient, V the magnitude of the filtration velocity.

Observe that (6) gives two filtration coefficients in the neighbourhood of the J-axis: $F'_+(J)$ when the speed begins to increase from zero, and $F'_-(J)$ when the speed vanishes after a maximum speed is reached. Figure 1 shows the general aspect for the motion in a definite point. The 'loading' curve is represented by the symbol $+$ but the 'unloading' one by $-$; the maximum value of J is denoted by X.

3. Boundary conditions

In order to solve equation (7) we must know the boundary conditions. If S is the frontier of the domain filled by the porous medium, we shall have, if we adopt the hypotheses of underground hydrodynamics, on a supply surface

$$h = \text{const.}, \tag{8}$$

and on an impervious surface

$$\partial h/\partial n = 0. \tag{9}$$

In the case of a medium homogeneous by parts, let i and j be the indices for two homogeneous neighbouring media and S_{ij} their common surface. Denoting $h_i = h(M)$ for $M \in D_i$, we shall have from the condition of the continuity of pressures

$$h_i = h_j \quad \text{on} \quad S_{ij}, \tag{10}$$

but from the condition of the continuity of normal velocities,

$$|\text{grad}\, h_i|^{-1} F_i(|\text{grad}\, h_i|)\, \partial h_i/\partial n = |\text{grad}\, h_j|^{-1} F_j(|\text{grad}\, h_j|)\, \partial h_j/\partial n; \tag{11}$$

here F_i and F_j are to be replaced by F_{i+} and F_{j+} when $dV/dt > 0$ and by F_{i-} and F_{j-} when $dV/dt < 0$.

As in the case of motions with initial gradient, here also domains where the fluid does not move can exist in porous media. Let D_0 be the sum of these domains and S_0 its frontier towards the domains where there is motion. In D_0 we have then a filtration fluid at rest and in this domain the function h_0 ($h_0 = h(M)$ for $M \in D_0$) must satisfy a partial differential equation which depends on the physical mechanism causing the inhibition of flow below the initial pressure gradient. At present this equation is not known, and for the time being it will be supposed to be Laplace's equation. The continuity of pressure on S_0 requires generally that

$$h_0 = h_i \quad \text{on} \quad S_0. \tag{12}$$

If h_0 satisfies equation (7) it goes without saying that this condition is fulfilled. We also have

$$|\text{grad}\, h| = a_{\pm} \quad \text{on} \quad S_0, \tag{13}$$

at points where D_0 is being eroded (+) or extended (−), although $|\text{grad}\, h|$ may sometimes have an intermediate value at points where S_0 is stationary.

An essential difference with respect to the cases previously studied (Gheorghitza 1959, 1961) is the possibility here of the existence of a domain D_+ different from D_-, i.e. the fluid may move in one domain when $dV/dt > 0$ and in another domain when $dV/dt < 0$.

4. Motions in homogeneous media

Let us write equation (7) when we have a homogeneous medium as

$$F_{\pm}(|\text{grad}\, h|) = A_{\pm}(|\text{grad}\, h| - a_{\pm})^{n_{\pm}}, \tag{14}$$

where A_+, A_-, a_+, a_-, n_+ and n_- are positive constants.

Let us suppose that on the positive part of the curve $V = F_+(|\text{grad}\, h|)$ we come slowly to the point $|\text{grad}\, h| = X$ and from that point we diminish $|\text{grad}\, h|$ steadily. Then between the constants A_+, A_-, a_+, a_-, n_+ and n_- we have the relation

$$A_+(X - a_+)^{n_+} = A_-(X - a_-)^{n_-}. \tag{15}$$

From (14) and (7) it follows that

$$2(|\text{grad}\, h| - a)\,\Delta h + [\{(n-1)\,|\text{grad}\, h| + a\}/|\text{grad}\, h|^2]\,\text{grad}\, h \,.\, \text{grad}\,(|\text{grad}\, h|^2) = 0, \tag{16}$$

where a must be replaced by a_+ and by a_-, respectively. Equation (16) becomes Darcy's equation when $a = 0$ and $n = 1$. But when $a = 0$ and $n \neq 1$, then the initial gradient vanishes, and we have for h the equation

$$2\,|\text{grad}\, h|^2\,\Delta h + (n-1)\,\text{grad}\, h \,.\, \text{grad}\,(|\text{grad}\, h|^2) = 0. \tag{17}$$

If $n = 1+s$, where $s \ll 1$, we can use the method of successive approximations, looking for h in the form

$$h = h_{(0)} + sh_{(1)} + s^2 h_{(2)} + \dots. \tag{18}$$

Substituting (18) into (17) for the functions $h_{(0)}, h_{(1)}, h_{(2)}, \dots$, we obtain the equations

$$\Delta h_{(0)} = 0,$$

$$2\,|\operatorname{grad} h_{(0)}|^2 \Delta h_{(1)} = -\operatorname{grad} h_{(0)} . \operatorname{grad}(|\operatorname{grad} h_{(0)}|^2),$$

$$2\,|\operatorname{grad} h_{(0)}|^2 \Delta h_{(2)} = 2\operatorname{grad} h_{(0)}[\operatorname{grad} h_{(0)} . \operatorname{grad}(|\operatorname{grad} h_{(0)}|^2)\operatorname{grad} h_{(1)}/|\operatorname{grad} h_{(0)}|^2$$
$$-\operatorname{grad}(\operatorname{grad} h_{(0)} . \operatorname{grad} h_{(1)})] - \operatorname{grad} h_{(1)} . \operatorname{grad}(|\operatorname{grad} h_{(0)}|^2),$$

....

The case of linear variation of V with $|\operatorname{grad} h|$, both when $dV/dt > 0$ and $dV/dt < 0$, is obtained as a particular case of (14) for $n_{\pm} = 1$:

$$F_{\pm}(|\operatorname{grad} h|) = A_{\pm}(|\operatorname{grad} h| - a_{\pm}). \tag{19}$$

With the same notation for X, we have instead of (15) the relation

$$A_{+}(X - a_{+}) = A_{-}(X - a_{-}), \tag{20}$$

from which it follows that if $a_{+} > a_{-}$, then $A_{+} > A_{-}$ and conversely, when $a_{+} < a_{-}$, then $A_{+} < A_{-}$; if $a_{+} = a_{-}$, then, evidently, $A_{+} = A_{-}$. The equation for h will be in this case

$$2(|\operatorname{grad} h| - a)\Delta h + a\,|\operatorname{grad} h|^{-2}[\operatorname{grad} h . \operatorname{grad}(|\operatorname{grad} h|^2)] = 0, \tag{21}$$

and this equation becomes the well-known equation $\Delta h = 0$ when $a_{\pm} = 0$.

5. Motions without D_0

Let us now consider some simple motions when D_0 is missing, $F_{\pm}$ is given by (19), and the solution of the problem can be written exactly.

(*a*) First consider the one-dimensional motion when we know the values of h at the ends of a layer of length L. Denoting $h(0) = h_{\mathrm{I}}$ and $h(L) = h_{\mathrm{II}}$, in order to fix our ideas, let us suppose that at the initial instant $h_{\mathrm{I}} = h_{\mathrm{II}}$, and that h_{I} then slowly increases but h_{II} is left unchanged (which amounts to saying that the pressure grows at one end of the layer, but the pressure at the other end is fixed). So long as $(h_{\mathrm{I}} - h_{\mathrm{II}})L^{-1} < a_{+}$, we have no motion in the porous medium, but when $(h_{\mathrm{I}} - h_{\mathrm{II}}) > a_{+}L$, motion begins to take place, the velocity being given by the relation

$$u = A_{+}[(h_{\mathrm{I}} - h_{\mathrm{II}})L^{-1} - a_{+}]. \tag{22}$$

Assume that h_{I} increases to a certain maximum value, and then, h_{II} being left constant, h_{I} starts to decrease. Then the velocity also begins to decrease, and it will be given by

$$u = A_{-}[(h_{\mathrm{I}} - h_{\mathrm{II}})L^{-1} - a_{-}], \tag{23}$$

for $(h_{\mathrm{I}} - h_{\mathrm{II}})L^{-1} > a_{-}$.

If we denote by S the cross-sectional area of the porous medium, normal to the direction of motion, and by T_{+} the time taken for h_{I} to increase from the value $h_{\mathrm{II}} + a_{+}L$ to the value XL, and by $T_{-} - T_{+}$ the time interval in which h_{I} decreases to $h_{\mathrm{II}} + a_{-}L$, we shall have a general formula for the volume of fluid which passes through the layer; denoting $(h_{\mathrm{I}}-h_{\mathrm{II}})\,L^{-1}$ by $G_{\pm}(t)$ according as to whether we are in the interval $(0, T_{+})$ or in the interval (T_{+}, T_{-}), this formula is

$$V = S\left[A_{+}\int_{0}^{T_{+}} G_{+}(t)\,dt + A_{-}\int_{T^{+}}^{T^{-}} G_{-}(t)\,dt + T_{+}(A_{-}a_{-} - A_{+}a_{+}) - A_{-}a_{-}T_{-}\right]. \tag{24}$$

(*b*) Another simple case is that of a plane motion having central symmetry. Let us denote $h_{\mathrm{I}} = h(r_{\mathrm{I}})$ and $h_{\mathrm{II}} = h(r_{\mathrm{II}})$, where $r_{\mathrm{II}} > r_{\mathrm{I}}$ and $h_{\mathrm{I}} > h_{\mathrm{II}}$ throughout the motion. Starting from the condition of incompressibility written in the form

$$q/2\pi r = -A(dh/dr + a), \tag{25}$$

where q is the discharge of fluid passing through unit thickness of the porous medium, we have at any instant

$$q_{\pm} = 2\pi A_{\pm}[h_{\mathrm{I}} - h_{\mathrm{II}} + a_{\pm}(r_{\mathrm{I}} - r_{\mathrm{II}})]\,[\ln(r_{\mathrm{II}}/r_{\mathrm{I}})]^{-1}, \tag{26}$$

$+$ corresponding to the period when $dV/dt > 0$ and $-$ to the period when $dV/dt < 0$. If we know the difference $h_{\mathrm{I}} - h_{\mathrm{II}}$ as a function of t, we have $q_{\pm} = q_{\pm}(t)$, and the whole volume of fluid which passes through the layer may be easily computed.

(*c*) For motion with spherical symmetry, denoting similarly $h_{\mathrm{I}} = h(r_{\mathrm{I}})$ and $h_{\mathrm{II}} = h(r_{\mathrm{II}})$, where $r_{\mathrm{II}} > r_{\mathrm{I}}$ and $h_{\mathrm{I}} > h_{\mathrm{II}}$, we start in a like manner from the relation

$$Q/4\pi r^2 = -A(dh/dr + a), \tag{27}$$

and after integration and determination of constants it follows that

$$Q_{\pm} = 4\pi A_{\pm}[h_{\mathrm{I}} - h_{\mathrm{II}} + a_{\pm}(r_{\mathrm{I}} - r_{\mathrm{II}})]\,(r_{\mathrm{I}}^{-1} - r_{\mathrm{II}}^{-1})^{-1}. \tag{28}$$

6. Example in which $D_{+} \neq D_{-}$

Let us consider now a simple example in which $D_{+} \neq D_{-}$. The porous medium fills the lower half-plane $0 > \theta > -\pi$ minus the half-circle $r < R$, and the radii $\theta = 0$ and $\theta = -\pi$ represent supply surfaces. The supply surface $\theta = 0$ is in contact with a basin B_1 and the other supply surface is in contact with a basin B_2. If $H(> 0)$ is the difference between the fluid levels in B_1 and B_2, then on $\theta = 0$ we can take $h = 0$ and then on $\theta = -\pi$ we have $h = H$. The specified half-circle being filled by an impervious medium, the boundary conditions satisfied by the function h are then

$$h = \begin{cases} 0 & \text{for} \quad r > R, \quad \theta = 0 \\ H & \text{for} \quad r > R, \quad \theta = -\pi \end{cases},$$

$$\partial h/\partial r = 0 \quad \text{for} \quad r = R,\ -\pi < \theta < 0.$$

The function h which satisfies these conditions and equation (21) is

$$h = -H\theta\pi^{-1}$$

and is a harmonic function. The fluid speed is

$$V = A[H(\pi r)^{-1} - a].$$

The fluid does not move during the first period in the domain in which

$$r > H(\pi a_+)^{-1} = R_+,$$

and in the period when $dV/dt < 0$, the fluid does not move for

$$r > H(\pi a_-)^{-1} = R_-,$$

and as we assumed that $a_- \neq a_+$, it follows that, for the same H, $R_+ \neq R_-$. Consequently, for the discharge we shall have the expressions

$$Q_+(t) = A_+[H\pi^{-1}\ln(R_+R^{-1}) - a_+(R_+ - R)],$$

and

$$Q_-(t) = A_-[H\pi^{-1}\ln(R_-R^{-1}) - a_-(R_- - R)].$$

In particular, if we suppose that H grows linearly with time, starting from the value zero, so that

$$H(t) = Nt,$$

it follows that the fluid begins to move only at the instant

$$t = T_0 = \pi R a_+ N^{-1}.$$

If the increase proceeds till the time $t = T_+$ ($> T_0$), at that instant the fluid moves in the domain $0 > \theta > -\pi$, $R_+ > r > R$, where

$$R_+ = NT_+(\pi a_+)^{-1}.$$

Supposing now that H decreases linearly from the maximum value reached to zero, i.e.

$$H(t) = NT_+ - M(t - T_+), \quad \text{for} \quad t > T_+,$$

H will vanish at the instant

$$T^* = T_+(NM^{-1} + 1),$$

but the motion will vanish before then, namely at the instant

$$t^* = T^* - \pi R a_-(M)^{-1}.$$

This example generalizes a simple motion of Newtonian filtration liquids (Polubarinova-Kotchina 1952, p. 219).

7. Concluding remarks

The above considerations are valid only for sufficiently small speeds, that is for speeds in the range corresponding to the linear motions of Newtonian filtration fluids. If the speed exceeds a certain value we shall have another relation between **V** and **J** instead of equation (6), and it is known that in this case the inertial terms are no longer negligible, and later the motion can become turbulent.

At present some points in the theory of visco-plastic filtration liquids are still obscure. For instance, we do not know the exact equation satisfied by h in D_0. Moreover, there are not enough suitable experimental investigations to establish the validity of one relation rather than of another; only careful experiments performed on the motion of visco-plastic fluids in porous media could establish the validity of the law (14) for $n \neq 1$ or of the law (19).

The results obtained here can be generalized. In the case of inhomogeneous media, when, for instance, A, a and n from (14) depend generally on position, one could obtain a quasi-linear partial differential equation with variable coefficients.

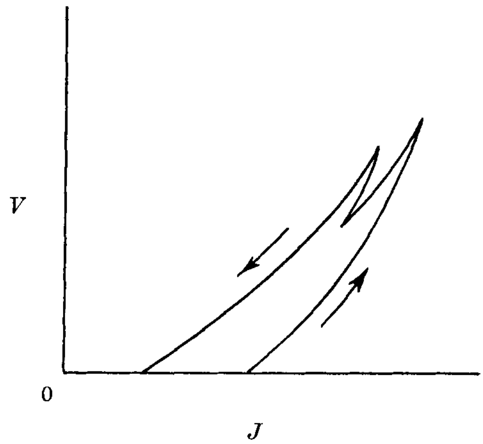

FIGURE 2. Effect of successive increases and decreases in gradient.

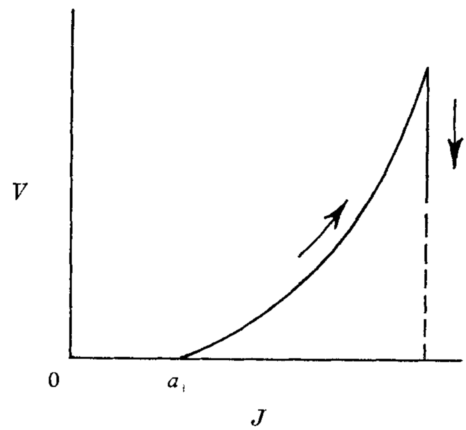

FIGURE 3. Possible behaviour of porous medium.

In this paper the compressibility of the fluid and of the porous medium were neglected. If these phenomena are taken into account, then one obtains for h an equation containing also a term with $\partial^2/\partial t^2$, such as T. Oroveanu and H. Pascal (Oroveanu 1963, p. 58) obtained for the linear motions of Newtonian fluids.

Visco-plastic fluids in porous media could exhibit other special features. It has been supposed above that the gradient of the pressure grows monotonically to a given value and then decreases monotonically. But if we have successive increases and decreases in the gradient of pressure, then at each decrease or increase yet another curve $V = F(|\text{grad}\, h|)$ is traced out, as pointed out in figure 2.

Lastly, let us mention that visco-plastic fluids could exhibit in some cases a variation of velocity, even when the pressure on the surface of the porous medium is constant (for instance, the case represented in figure 3).

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